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The Council transacted further business in connection with the MONTHLY and with the annual meeting of the Association. In view of the developments at the Dartmouth meeting it seemed inevitable and of the greatest importance that the subject of mathematics courses for the period of the war should form the predominating feature of the December program. Whether the Association should meet in affiliation with the American Mathematical Society, whether in affiliation with the American Association for the Advancement of Science at Baltimore, whether the Association will best serve the interests of its members by holding one meeting or, instead of this, enable more members to be within reach of the meetings by having duplicate programs in the East and in the West, were the subjects of earnest discussion. By the time this report appears it will doubtless have been decided by the full participation of the Council as to what place is wisest.

W. D. CAIRNS, *Secretary-Treasurer.*

MATHEMATICAL ENCYCLOPEDIA DICTIONARY.

By G. A. MILLER, University of Illinois.

The following preliminary article is intended to serve as a basis for discussions relating to the nature and the extent of the major articles in the proposed mathematical dictionary. It aims to explain the terms *group* and *group theory* and to define a few of the most important terms which are related thereto. The latter terms should probably appear in their regular alphabetical places with references to the words *group* or *group theory* for their special meanings in this connection. In some cases this special meaning could not be made clear without such preliminary general developments as are here presented.

The object has been to give only such information as is within the range of the first-year graduate student, since the proposed dictionary should clearly not aim

at completeness but should confine itself to rendering very efficient service to certain classes of students. It is evidently a difficult matter to select those elements of a large subject which a student may be supposed to know at a certain stage of his development, in view of the great differences in interest and preparation. On the other hand, brief expositions which aim to present only what is most fundamental have a peculiar charm even for the mature students.

While the chairman of the "Dictionary Committee" of the Association was consulted as regards the desirability of publishing a preliminary article, he did not see the present article itself in manuscript, so that the committee is in no way responsible for its style or content. [See note on page 428. Ed.] The writer believes that the major mathematical terms should appear in five languages, English, French, German, Italian and Spanish, and hence the term group appears in these languages, in order, at the beginning of the article.

Group (grōop), *groupe*, *Gruppe*, *gruppo*, *grupo*. This term is frequently used in mathematics with its non-technical meaning, denoting a collection composed of a finite or of an infinite number of elements; *e. g.*, a group of terms, a group of points, a group of waves, group insurance, etc. The most general technical mathematical meaning of the word group is obtained by restricting its non-technical meaning by imposing the condition that each of the elements of the collection can be combined with itself and also with every other element of the collection. Moreover, the result obtained by such a combination must also be an element of the collection. This most general technical meaning of the term group includes all its other technical meanings as well as that of domain of rationality. It appears frequently in the geometrical literature, but it occurs also in the literature on analysis. Cf. *Encyclopédie des Sciences Mathématiques*, tome I, volume 2, page 243.

The most common additional restriction imposed on the elements of a group is that the *inverse* of each element shall also be contained in the collection; *i. e.*, each element of the collection can be combined with at least one element of the collection so as to obtain the *identity*. Among the most common additional restrictions are the following: when the elements are combined, or *multiplied together*, the associative law holds, and if any two of the symbols in the equation

$$xy = z$$

are replaced by elements of the collection the equation has always one and only one root in the collection.

All these restrictions are usually imposed on the elements of a finite collection representing a mathematical group, but frequent exceptions appear when the collection contains an infinite number of elements. Cf. Lie, *Theorie der Transformationsgruppen*, volume 1, page 163, where an infinite group which does not involve

the inverse of any one of its elements is discussed. Some writers contend that even an infinite collection does not represent a group unless its elements satisfy all the conditions of combination noted above. Cf. A. Loewy, *Archiv der Mathematik und Physik*, volume 9, (1905), page 105, where it is stated that the definition given on page 218, volume 1, *Encyclopédie der Mathematischen Wissenschaften*, is incorrect as regards infinite groups, because it is satisfied by a collection of elements which does not include the inverses of its elements. A group in the most restrictive sense of the term has been called *ordinary group* by L. Autonne, Paris, *Comptes Rendus*, vol. 143 (1906), p. 671.

E. Galois (1811-1832) seems to have first used the term group with a technical mathematical meaning. In fact, Galois and the other writers before the middle of the nineteenth century practically confined their group theory studies to finite groups whose elements are represented by substitutions, and every finite collection of substitutions satisfies the most restrictive definition of group noted above provided it satisfies the most general technical definition. Difficulties as regards the most desirable definitions of the technical term group began to appear when A. Cayley inaugurated the study of abstract groups (1854), and C. Jordan exhibited the wide usefulness of infinite groups (1868). These difficulties cannot be regarded as solved at the present time.

In a broad way groups have been divided into four categories, as follows: Finite discontinuous, infinite discontinuous, finite continuous and infinite continuous. The first of these categories of groups began to be studied during the latter half of the eighteenth century in connection with the solution of algebraic equations in one unknown, and the terminology thus developed was largely transferred to similar concepts arising in connection with the study of the other categories. Among the important concepts common to all of these

categories are the following: subgroup, invariant subgroup, conjugate groups and quotient group. It may be noted that some groups which are continuous in the terminology of S. Lie (1842-1899) have been called discontinuous by H. Taber, *Bulletin of the American Mathematical Society*, vol. 6 (1900), p. 202.

An illustration of a group in the most general mathematical sense is furnished by the natural numbers when they are combined by addition or multiplication, or by both of these operations separately. When the positive and negative integers together with 0 are combined by addition there results a group in the most restricted sense. Both of these are *infinite discontinuous* groups. An instance of a *finite discontinuous* group is furnished by the 24 movements of space which transform a cube into itself, while the totality of the movements of space which transform a point into itself constitutes an illustration of a *finite continuous* group. The elements of a continuous group are not denumerable. Such a group is composed of one or several families of transformations each depending upon a finite number of parameters.

An infinite continuous group contains an infinite number of parameters, or one or more parametric functions. The theory of infinite continuous groups has been less developed than that of finite continuous groups. In all cases except perhaps that of finite discontinuous groups there is as yet no uniformity of usage as regards the essence of the restrictions to be imposed on the most general technical definition of a group although the restrictions imposed by S. Lie have been very widely adopted in works on continuous groups. It should be emphasized that a collection alone is not a technical group. There must be also some law of combination.

The concept involved in the most general technical definition of the term group is that of a system of elements which is *closed* as regards one or more than one method of combining these elements. This concept is as old as mathematics itself since it appears in the fundamental operations of arithmetic and it also underlies Euclidean geometry. Cf. H. Poincaré, *The Monist*, volume 9, 1899, where it is stated, on page 31, that without the notion of group there would be no geometry. In fact, it appears probable that the successive extensions of the number concept, so as to include fractions, and negative, irrational and complex numbers, was largely due to the fact that the most general technical notion of group is ingrained in us and has influenced our intellectual development. Notwithstanding the great importance of this general technical notion and the need of a special name for it, it is too general to serve as a basis of a special abstract theory at the present stage of mathematical development.

Efforts to formulate an abstract definition of group, sufficiently special to serve as a basis

for an autonomous theory, were inaugurated after the most general technical notion of group had been extensively applied to a set of concrete elements (substitutions) which intrinsically obey the associative but not the commutative law of combination. The theory of substitutions, more than the theory of numbers, thus furnished the model for abstract definitions of a group, which were first clearly formulated by H. Weber (1882), and G. Frobenius (1887). Somewhat earlier (1870) L. Kronecker had formulated such a definition for an abstract abelian group. Among later formulations of such definitions we would refer especially to those which appeared in the *Transactions of the American Mathematical Society* and are due to E. H. Moore (1902, 1904, 1905), L. E. Dickson (1905), E. V. Huntington (1903, 1905, 1906). Cf. W. A. Hurwitz, *Annals of Mathematics* (1906-1907), page 94.

An explicit abstract formulation of a definition of group in the most general technical sense, including both continuous and discontinuous groups, was given by S. Lie in 1871, *Forhandlinger Videnskabs-Selskabet*, edited in 1872, page 243; but in the development of his theory of continuous groups Lie imposed additional restrictions so that his groups can be defined by means of differential equations. In the preface to volume 3 of his *Theorie der Transformationsgruppen* (1888-1893), which aims to give a pure abstract theory of finite continuous groups, he directed attention (p. 17) to the fact that F. Klein used the term continuous group in a more general sense than that adopted by Lie.

Among the other terms used for group are the following: *Permutation* (Ruffini), *system of conjugate substitutions* or *conjugate system* (Cauchy), *closed system* (Lie and Klein in their earliest publication only). The term employed by Cauchy has been extensively used by others, especially by French writers. In particular, it is used instead of group in the *Cours d'algèbre supérieure* by J. A. Serret, being retained in the sixth edition (1910). When a part of the elements of a group constitute a group the latter is called a *subgroup* or a *divisor* of the former.

When the elements of a group are symbols of operation having no intrinsic properties the group is said to be *abstract*; when these symbols have intrinsic properties the group is said to be *concrete*. Abstract groups are sometimes called *general groups*. It is therefore necessary to distinguish between general groups and groups satisfying the most general technical definition of this term. When the elements of a group combine according to the commutative law the group is said to be *commutative* or *abelian*. When the elements of a group represent geometric concepts the group is called a *geometric group*. Groups which are not geometric are usually called *algebraic* or *analytic*. When the elements of a group are represented

by substitutions it is called a *substitution group*.

If s_1 and s_2 represent two elements of a group G the element $s_1^{-1} s_2 s_1$ is called the *transform* of s_2 with respect to s_1 . When s_2 remains fixed while s_1 is replaced successively by all the elements of G there results a totality of elements known as the *complete set of conjugates* of s_2 under G . The elements of this set when they are combined in every possible manner generate a group which includes the complete set of conjugates of all its elements under G . When this group is not identical with G it is said to be an *invariant* or *self-conjugate subgroup* of G . An invariant subgroup is characterized by the fact that it includes the complete sets of conjugates of each of its elements, and when a group does not contain any such subgroup besides the identity it is called a *simple group*. A group which is not simple is said to be *composite*. Invariant subgroups are also called *normal divisors*, *proper divisors*, *monotypic*, *self-conjugate*, etc.

When all the elements of a group H are transformed by an element s of H , or of a larger group G in which H appears as a subgroup, there results a group H' known as the *conjugate* of H with respect to s . It results directly that H is, in turn, the conjugate of H' with respect to the inverse of s , or s^{-1} . When s represents successively all the elements of G there results a *complete set of conjugates* of H under G . A necessary and sufficient condition that H is invariant under G is that all of these conjugates are identically equal to each other. All the elements of G can be divided with respect to H into sets (known as *co-sets*), and when H is invariant these sets combine as units and constitute a group called the *quotient group* of G with respect to H , which is denoted by G/H . A quotient group is also called a *factor group*, or a *complementary group*.

The elements of an abstract group are often called *operators* or *operations*. When the number of these operators is finite it is called the *order* of the group. On the other hand, the order of a finite continuous group is the number of its arbitrary parameters. A substitution group on n letters is said to be of *degree* n . Some writers use order for degree and vice versa. A substitution group is said to be *regular* when its order is equal to its degree and every substitution besides the identity involves all the letters.

When each letter of a substitution group is replaced by a particular letter in some substitution of the group it is called a *transitive group*. A regular group is necessarily transitive. When a substitution group is not transitive it is said to be *intransitive*. A transitive substitution group whose letters can be divided into sets such that every substitution of the group transforms all the letters of each of these sets either among themselves or into those of another one of these sets is called

imprimitive or *non-primitive*. All transitive groups which are not imprimitive are said to be *primitive*. While the term primitive is positive it is usually defined negatively and the converse is true as regards the term imprimitive.

It is important to note that in the theory of continuous groups the intransitive groups are classed with the imprimitive groups while this is not done in the case of finite substitution groups. A substitution group which involves all the possible substitutions on its letters is called *symmetric*. The symmetric group of degree n is of order $n!$. The substitution group composed of exactly half the substitutions of a symmetric group is called *alternating*. When a substitution group has the property that it replaces a sub-set of r of the letters by every possible such set of its letters it is said to be *r-times* or *r-fold transitive*. The alternating group of degree n is $(n-2)$ -fold transitive while the symmetric group of this degree may be said to be either $(n-1)$ -fold or n -fold transitive.

There are groups which are composed of a finite number of families of continuous transformations. For instance, the movements of the plane which transform a point into itself and are represented by the following two sets of equations:

$$x' = x \cos \theta - y \sin \theta,$$

$$y' = x \sin \theta + y \cos \theta,$$

and

$$x' = x \cos \theta + y \sin \theta,$$

$$y' = x \sin \theta - y \cos \theta.$$

These two families of continuous transformations constitute a group, but one cannot pass continuously from a transformation of the former type to one of the latter since the determinants of the former are equal to unity while those of the latter are equal to -1 . Such groups are sometimes called *mixed* or *complex* groups.

There is still a considerable lack of uniformity as regards the use of terms in group theory. For instance, the term *principal group* (Hauptgruppe) was used by G. Frobenius (*Crelle*, vol. 86, 1879, p. 219) to represent the group formed by the identical element, while F. Klein employed the same term for the continuous mixed group of movements whose invariants constitute elementary geometry. Similarly, the term *anharmonic group* has been employed for two distinct groups; viz., for a group of order 6 which obeys the same laws of combination as the symmetric group of degree 3, and hence is said to be *simply isomorphic* with this group (Pascal's *Repertorium*, vol. 1, 1910, p. 238), and also for the non-cyclic group of order 4, commonly known as the *four-group* (Capelli, *Istituzioni di analisi algebrica*, 1909, p. 111).

Group theory. The systematic development of theorems relating to properties of

groups is known as group theory. Among the various branches of this science the theory of substitution groups is the oldest, having been founded by A. L. Cauchy, about 1845, and first embodied in the form of a separate treatise by C. Jordan, *Traité des substitutions et équations algébriques*, 1870.

The theory of substitutions contains two large branches. The older of these is sometimes called the theory of permutation groups, and is based on the possible interchanges of letters, while the other branch is involved in the theory of linear transformations, and is commonly known as the theory of linear substitution groups.

The theory of finite abstract groups is intimately connected with the two theories of substitution groups just noted and was first embodied in the form of a separate treatise by W. Burnside, *Theory of groups of finite order*, 1897; second edition with greater emphasis on linear groups, 1911. These three theories are sometimes referred to as *algebraic group theory*. There is, however, no clear line of distinction between algebraic group theory and the group theories of analysis and geometry.

The group theory of analysis may also be divided into three large branches, viz., theory of finite continuous groups, theory of infinite continuous groups, and theory of groups of automorphic functions. The first of these theories was first developed in a systematic manner by S. Lie, *Theorie der Transformationsgruppen*, three large volumes, while the last was treated in volume 1 of *Automorphe Functionen* by R. Fricke and F. Klein, 1897. No systematic treatise on the general theory of infinite continuous groups has yet been published.

Geometric group theory is based on the group theories of algebra and analysis. In geometry the group concept has entered more widely into the various developments than in algebra or in analysis. Among the treatises devoting considerable space to geometric groups we may mention Klein's *Einleitung in die höhere Geometrie*, II, 1893, and Lie's *Geometrie der Berührungstransformationen*, 1896.

C. Alasia prepared a general bibliography on group theory, which was published in volumes 18-22 of *Rivista di fisica matematica e scienze naturali*, Pavia. A bibliography relating to finite groups together with many historical data may be found in the *Constructive development of group theory* by B. S. Easton, 1902. Among the treatises on the theory of groups which were not noted above are the following: E. Netto, *Substitutionentheorie*, 1882; translated into Italian by G. Battaglini, 1885, and into English by F. N. Cole, 1892; S. Lie and G. Scheffers, *Vorlesungen über Differentialgleichungen*, 1891, and *Vorlesungen über kontinuierliche Gruppen*, 1893; G. Vivanti, *Teoria dei gruppi di trasformazioni*, 1898; translated into French by A. Boulanger, 1904; L. Bianchi, *Lezioni sulla teoria dei gruppi di sostituzioni*, 1900; L. E. Dickson, *Linear Groups*, 1901; J. E. Campbell, *Theory of Continuous Groups*, 1903; J. A. de Séguier, *Groupes Abstraites*, 1904; G. Fubini, *Teoria dei gruppi discontinui e delle funzioni automorfe*, 1908; H. Hilton, *Finite Groups*, 1908; E. Netto *Gruppen- und Substitutionentheorie*, 1908; J. A. de Séguier, *Groupes de Substitutions*, 1912; Miller, Blichfeldt and Dickson, *Theory and Applications of Finite Groups*, 1916; H. F. Blichfeldt, *Finite Collineation Groups*, 1917.

FUNDAMENTALS IN THE MATHEMATICS OF INVESTMENT.

By EDWARD LEWIS DODD, University of Texas.

§ 1. INTRODUCTION.

In most books on the mathematics of investment there is a wealth of formula somewhat forbidding to the casual reader, however necessary it may be to the accountant or actuary. It is the object of this paper to present in rather compact form some of the fundamentals of the subject, with a few general formulas of wide application.

§ 2. INTEREST AND DISCOUNT.

The mathematics of investment deals with the *increment of value*. If P units of value—say P dollars—at one moment of time are worth or conceived to be worth S units at a later moment, the increment $S - P$ is called the *interest*¹

¹ For the general theory, it is immaterial whether the change of value is brought about by a loan or by a series of commercial transactions, indeed, whether the increment is positive or negative.